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Motion of a scalar field coupled to a Yang-Mills field reformulated locally with some gauge invariant variables

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Abstract: This paper exposes a reformulation of some gauge theories in terms of explicitly gauge-invariant variables. We show in the case of Scalar QED that the classical theory can be reformulated locally with some gauge invariant variables. We discuss the form of some realistic asymptotic solutions to these equations. The equations of motion are then also reformulated in the non-abelian case.

Keywords: Solitons Monopoles and Instantons, Gauge Symmetry.

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1. Introduction

The gauge symmetry is known to render the calculation of the elements of the S matrix very intricate. In some future colliders like LHC or NLC, some very complicated scattering processes will be studied. Phenomenologists will have to consider processes with 3, 4 or more particles in the final state. The scattering amplitudes for these processes are in general very complicated because of the very large number of Feynman graphs, and the numerical evaluation of these amplitudes in Monte-Carlo programs suffer from numerical instabilities due for a large part to some huge compensations between the different graphs, which arise from the gauge symmetry. To avoid these numerical instabilities, there are two common methods. The first one consists in using a specific gauge which simplifies the different vertices and propagators [1, 2, 3, 4, 5]. The second one consists in using some algorithms acting on each Feynman graph, based on Ward identities, in order to simplify the expression of the graphs [6]. Both methods lead to the elimination of most of these huge compensations.

In this paper, we consider this problem from another point of view, at the core of Quantum Field Theory. Basically, we raise the question of whether the calculations of the elements of the S matrix can be done directly using some gauge invariant variables. This question can be studied in the context of both methods cited above, using as a fundamental tool the Ward identities. These identities depend on the gauge fixing procedure used in the calculations. We rather look here for a method in which there is no need to break temporarily the gauge symmetry. As a consequence, we must start our formulation from the very beginning of gauge theories, that is to say from the equations of motion. We therefore show in this paper that one can reformulate these equations in terms of local gauge invariant variables for the case where matter fields are scalar.

This new approach may have some interesting applications regarding the quantization of fields. In standard field theory, the quantization procedure is done first on free fields, and therefore matter fields and gauge fields are considered separately, though they are coupled in the equations of motion. A significant consequence is that it is irrelevant to consider the evolution of a free field from a time t to an interacting field at time $t' > t$ through a unitary transformation, because Haag's theorem says that the field considered at time t' must be also free (for a good review, see [7]). Quantum Field Theory is therefore doomed to describe only the transition between asymptotic fields through the LSZ formalism. In experiments where the time variable plays a fundamental role (CP violation experiments in K^0_S/K^0_L, neutrino oscillations, ...) one must use a mixed theory, based in part on classical quantum mechanics (Rabi precession, ...) and in part on quantum field theory for the computation of the decay width of the particles. A single theory which would describe completely such experiments is still missing. Since Haag's theorem does not apply to the case of two constantly interacting fields, the approach presented in this paper opens the prospect of finding an evolution operator between two finite times for an interacting system. That is to say, an asymptotic electron would be described both by its matter field and its surrounding electromagnetic field, in some sense. So we must also find some "realistic" asymptotic solutions to the coupled equations of motion in replacement of the plane waves that are used in standard quantum field theory. The word "realistic" means here that we look for solutions that have a finite conserved momentum. We show that solitons are not possible in this context (for the U(1) case), but we conjecture that some periodic-in-time solutions may probably exist.

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What is the basic idea of our approach? We know that for a given field-strength tensor, one can compute a corresponding gauge field using the basic cohomological formulas that are reviewed in the appendices. Some authors have already tried to reformulate the Yang-Mills Theory using only the Field-Strength tensor as a basic variable in place of the gauge field [8, 9, 10, 11]. The results of these studies are generally not covariant and non-local, due to the fact that the cohomological formulas are essentially of a non-local nature. In this paper, we rather consider the gauge-currents as fundamental variables, and we keep both locality and covariance of the equations.

The paper is therefore organized as follows:

The section 2 is devoted to the reformulation of U(1) scalar QED in terms of gauge invariant variables.

The section 3 contains a discussion on asymptotic solutions of the U(1) scalar QED. We first show that periodic solutions of Klein Gordon do not have a finite energy, contrary to what is claimed in a recent paper, and therefore we need to consider the coupled equations. We show the impossibility of soliton solutions in this context, and discuss the possibility of periodic (in time) solutions.

The section 4 presents the non-abelian case, where the gauge group is in a certain class of subgroups of U(N). It turns out that the results presented in this paper are in fact a simpler version of some results given by Lunev in 1994 [12], with in addition the coupling to a scalar multiplet (he only considered a pure Yang-Mills theory).

2. A possible reformulation of classical SQED

In this section, we reformulate the classical theory of a scalar field coupled to a U(1) gauge field (SQED) in terms of gauge invariant variables. We will then demonstrate the difficulties appearing when one wants to find some "realistic" asymptotic solutions, which would generate a Fock-like space. Using such a space, one could then construct a new formalism for computing cross sections. Let us start with the classical scalar QED lagrangian:

$$\mathcal{L} = (D_\mu\phi)^* D^\mu\phi - m^2\phi^*\phi - \frac{1}{2}\partial_\mu A_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu)\,, \tag{2.1}$$

with $D_\mu = \partial_\mu + ieA_\mu$. The electrical current is given by $J_\mu = ie(\phi^*(D_\mu\phi) - (D_\mu\phi)^*\phi)$ and the probability density $\rho = \phi^*\phi$. Both J_μ and ρ are gauge invariant variables and we will show how to rewrite the previous lagrangian as a function of these variables (this treatment will have to be modified in the non-abelian case in which the corresponding expression for these variables are not gauge invariant but gauge "covariant"). First, we shall review the standard equations of motion when ϕ and

A^μ are taken as field variables:

$$0 = (D_\mu D^\mu + m^2)\phi \tag{2.2}$$
$$0 = (\Box + m^2)\phi + 2ieA_\mu\partial^\mu\phi + ie(\partial\cdot A)\phi - e^2(A\cdot A)\phi \tag{2.3}$$
$$\partial^\alpha F_{\alpha\beta} = ie(\phi^*(D_\beta\phi) - (D_\beta\phi)^*\phi) = J_\beta \tag{2.4}$$

We shall first note that if one computes $\phi^*(2.3) - (2.3)^*\phi$, one obtains $\partial_\mu J^\mu = 0$, which we would have already obtained by taking the divergence of eq. (2.4). The redundancy between the last two equations can therefore be removed by making use of $\phi^*(2.3) + (2.3)^*\phi$ instead of eq. (2.3). After some algebra, it is not a hard task to make J_μ and ρ appear in the equations as we will see later, but for the derivation of the new equations, we rather choose to start from the lagrangian. For this purpose, we will use the following relations:

$$-\frac{J^2}{e^2} = (\phi^*(D_\mu\phi) + (D_\mu\phi)^*\phi)^2 - 4(D_\mu\phi)^*\phi\phi^*(D_\mu\phi) \tag{2.5}$$
$$\Rightarrow (D_\mu\phi)^* D^\mu\phi = \frac{1}{4\rho}\left((\partial_\mu\rho)^2 + \frac{J^2}{e^2}\right) \tag{2.6}$$
$$= (\partial_\mu\sqrt{\rho})^2 + \frac{J^2}{4e^2\rho} \tag{2.7}$$

Throughout the paper, we will conveniently define v^μ such that $J^\mu = 2e^2\rho v^\mu$ and set $z(x) = \sqrt{\rho(x)}$. From the definition of the current, one can also extract the expression of the field strength tensor:

$$2e^2 F_{\mu\nu} = \partial_\mu\left(\frac{J_\nu}{\rho}\right) - \partial_\nu\left(\frac{J_\mu}{\rho}\right) \tag{2.8}$$
$$F_{\mu\nu} = \partial_\mu(v_\nu) - \partial_\nu(v_\mu) \tag{2.9}$$

Equations (2.7) and (2.9) are the fundamental tools of our formalism. With these, we can write the lagrangian as a function of z and v^ν in the following way:

$$\mathcal{L} = (\partial_\mu z)^2 - m^2 z^2 + e^2 z^2 v^2 - \frac{1}{4}(\partial_\mu(v_\nu) - \partial_\nu(v_\mu))^2 \tag{2.10}$$

We have now re-expressed the lagrangian in terms of gauge invariant quantities, and as a by-product the "effective" coupling constant is $e^2 = 4\pi\alpha$ instead of e. This means that the sign of e is not relevant. Although this does not mean that in a perturbative expansion of some solutions, the relevant expansion parameter is necessarily e^2, it may also be $\sqrt{4\pi\alpha}$ or $|e|$. From this new lagrangian we can derive the following equations of motion thanks to the Euler-Lagrange equations:

$$(\Box + m^2)z = 4\pi\alpha z v^2 \tag{2.11}$$
$$\Box(v_\nu) - \partial_\nu(\partial\cdot v) = -8\pi\alpha z^2 v_\nu \tag{2.12}$$

2.1 The energy-momentum tensor

We will further look for asymptotic solutions to the coupled equations with a finite conserved momentum. The symmetrized energy momentum tensor (or Belinfante tensor) can be rewritten this way:

$$\begin{aligned} T_{\mu\nu} &\stackrel{\text{def}}{=} (D_\mu\phi)^\dagger D_\nu\phi + (D_\nu\phi)^\dagger D_\mu\phi - F_{\mu\lambda}F_\nu{}^\lambda - \\ &- g_{\mu\nu}\left((D_\mu\phi)^\dagger D^\mu\phi - m^2\phi^\dagger\phi - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right) \qquad (2.13) \\ T_{\mu\nu} &= 2e^2z^2v_\mu v_\nu + 2\partial_\mu z\partial_\nu z - (\partial_\mu(v_\lambda) - \partial_\lambda(v_\mu))\left(\partial_\nu(v^\lambda) - \partial^\lambda(v_\nu)\right) - \\ &- g_{\mu\nu}\left((\partial_\mu z)^2 - m^2z^2 + e^2z^2v^2 - \frac{1}{4}(\partial_\mu(v_\nu) - \partial_\nu(v_\mu))^2\right) \qquad (2.14) \\ P_\mu &= \int_\Sigma d\sigma^\nu T_{\nu\mu}\,. \qquad (2.15) \end{aligned}$$

In eq. (2.15), Σ represents any space-like hyper-surface in the Minkowsky space time, and for the sake of simplicity, we will generally take the $t = 0$ hypersurface for the computation of P_μ.

3. Solitons solutions are not normalizable

In general, the spatial extent of the wave function of a free particle (obeying the Klein Gordon equation) increases in time. Here, we will rather look for the possibility to find "soliton-like" solution to the coupled field equations of scalar QED (i.e. eq. (2.11) and eq. (2.12)). First, we will show that for the Klein Gordon equation, we can find some simple "soliton-like" solutions but these solutions are not normalizable (similarly to plane waves). We obtain the same result when the interaction is taken into account, but the arguments used to reject this case are different from the free case. For this reason, the free case is also presented, even if it can be seen as a particular case of the interacting one.

3.1 Generalities about solitons

We will say that a function $f(x)$ defined on space-time is a soliton if we can find a time-like momentum p^μ such that:

$$p^\mu\partial_\mu f = 0\,. \qquad (3.1)$$

This time-like momentum represents the global momentum of the wave which moves without deformation. To see this trivial fact, eq. (3.1) simply means that if we are placed in a frame where $p^\mu = (m_0, \vec{0})$, then the shape of the wave function does not depend on time ($\partial_0 f = 0$). Suppose now that at time $t = 0$ we look at the shape of the wave function. It is reasonable to say that for an asymptotic solution (supposed to describe a free scalar particle) the probability density is spherically symmetric. We can deduce from that that the function f is a function of only one variable. To be more specific, let us consider the two variables $u = (p \cdot x)^2 - p^2 x^2$ and $\tau = p \cdot x$. In the "rest frame", where $p^\mu = (m_0, \vec{0})$,[1] then:

$$\begin{aligned} u &= (p \cdot x)^2 - p^2 x^2 \\ &= (m_0 t)^2 - m_0^2 (t^2 - \vec{x}^2) = m_0^2 \vec{x}^2 \\ \tau &= p \cdot x = m_0 t \,. \end{aligned} \tag{3.2}$$

A "spherically symmetric" scalar function f is therefore a function of u and τ only. We have seen that $u \geq 0$ for any x and we will often write $y = \sqrt{u}$. We have by construction $u(x^\mu + \lambda p^\mu) = u(x^\mu)$, which means that a function of the variable u is invariant under any translation in the p^μ direction. For convenience, we will also use the following notations:

$$\begin{aligned} \lambda^\alpha &= p^\alpha (p \cdot x) - p^2 x^\alpha = \frac{1}{2} \partial^\alpha u(x) \\ \lambda^2 &= -p^2 u(x) = -m_0^2 u(x) \qquad p \cdot \lambda = 0 \\ \partial_\alpha \lambda_\beta &= p_\alpha p_\beta - m_0^2 g_{\alpha\beta} = \tau_{\alpha\beta} \\ \partial^\alpha \lambda_\alpha &= \tau^\alpha_\alpha = -3 m_0^2 \\ p^\alpha \tau_{\alpha\beta} &= 0 \qquad \lambda^\alpha \tau_{\alpha\beta} = -m_0^2 \lambda_\beta \\ \partial_\mu \sqrt{u} &= \frac{\lambda_\mu}{\sqrt{u}} \qquad \partial^\mu \left(\frac{\lambda_\mu}{\sqrt{u}} \right) = -\frac{2m_0^2}{\sqrt{u}} \,. \end{aligned} \tag{3.3}$$

And in the rest frame, $\lambda^\mu = -m_0^2 (0, \vec{x})$. Then, if the scalar function f is a "spherically symmetric" soliton, we have:

$$\begin{aligned} f(x) &= g(\tau, u = (p \cdot x)^2 - p^2 x^2) \\ 0 = p^\mu \partial_\mu f &= p^\mu (p_\mu \partial_0 g + 2 \lambda_\mu \partial_1 g) = p^2 \partial_0 g \,. \end{aligned} \tag{3.4}$$

Therefore g does not depend on τ, but only on u. We therefore obtain a covariant formulation of the notion of a "spherically symmetric" soliton.

3.2 Periodic solutions to the Klein Gordon equation

Before we look for some asymptotic solutions to the coupled equations, we must explain why we cannot have some realistic asymptotic states in the free case. Of course finite energy solutions to the Klein-Gordon equation exist, consisting in wave-packets with square integrable momentum densities. But one of the criteria we set in order to define a "realistic" asymptotic field is that the wave-packet must be "bounded" in space-like directions. We consider here that a constantly spreading

[1] We shall remark that $p^2 = m_0^2$, where the mass m_0 is *a priori* different from the mass m appearing in the Klein Gordon equation.

wave-packet cannot represent the state of a stable free particle. We show in this paragraph that soliton solutions to the Klein-Gordon equation cannot have a finite energy, as a particular case of a stronger result concerning periodic-in-time solutions. The free scalar lagrangian is:

$$\mathcal{L} = \partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi \,. \tag{3.5}$$

The energy-momentum tensor and the corresponding conserved total momentum are:

$$\begin{aligned} T_{\mu\nu} &= \partial_\mu \Phi^* \partial_\nu \Phi + \partial_\nu \Phi^* \partial_\mu \Phi - \mathcal{L} g_{\mu\nu} \\ P_\nu &= \int_\Sigma d\sigma^\mu T_{\mu\nu} \,. \end{aligned} \tag{3.6}$$

The linearity of the equations of motion allows us to expand the field in a Fourier series:

$$\begin{aligned} \phi(t,\vec{x}) &= \Sigma_{n=-\infty}^{\infty} a_n(\vec{x}) e^{in\omega t} \\ &= \Sigma_{n=-\infty}^{\infty} a_n(r) e^{in\omega t} \,. \end{aligned} \tag{3.7}$$

We first look for solutions of the form $\phi = \exp(i\eta p \cdot x) g(\sqrt{u})$, where η is a real parameter. We have:

$$\begin{aligned} \partial^\mu \phi &= e^{i\eta p\cdot x} \left(i\eta g(\sqrt{u}) p^\mu + \frac{\lambda^\mu}{\sqrt{u}} g'(\sqrt{u}) \right) \\ \Box \phi &= e^{i\eta p\cdot x} \left(-m_0^2 \eta^2 g(\sqrt{u}) - \frac{2m_0^2}{\sqrt{u}} g'(\sqrt{u}) - m_0^2 g''(\sqrt{u}) \right) \\ &= -\frac{m_0^2}{y} e^{i\eta p\cdot x} \left(\eta^2 t(y) + t''(y) \right) \qquad (g(y) = t(y)/y) \\ 0 &= (\Box + m^2)\Phi \\ \Leftrightarrow 0 &= t''(y) + t(y) \left(\eta^2 - \frac{m^2}{m_0^2} \right) \\ \Rightarrow t(y) &= A e^{-\sqrt{m^2/m_0^2 - \eta^2}\, y} \,. \end{aligned} \tag{3.8}$$

We have not considered the other solution that increases as y (or r) increases, because we look for normalized solutions. Thus, the general solution is:

$$\begin{aligned} \phi(x) &= \frac{1}{y} \sum_{|n| \leq \left[\frac{m}{m_0}\right]} A_n e^{inp\cdot x} e^{-\sqrt{\frac{m^2}{m_0^2} - n^2} y} \\ &= \frac{1}{m_0 r} \sum_{|n| \leq \left[\frac{m}{m_0}\right]} A_n e^{inm_0 t} e^{-\sqrt{m^2 - n^2 m_0^2} r} \,. \end{aligned} \tag{3.9}$$

The sum has a finite number of terms because we limit ourselves to exponentially decreasing terms. It will be clear in the following that the oscillating solutions for $|n| > [m/m_0]$ will not provide normalizable solutions. Contrary to the claim of Hormuzdiar and Hsu in [13] which considered only the large r behaviour, the solutions are not normalizable. This is due to their small r behaviour. This can be shown by computing the conserved momentum:

$$\begin{aligned} P_0 &= \int_{t=0} d\vec{x} \left[2\partial_0\phi^*\partial_0\phi - g_{00}(\partial_0\phi^*\partial_0\phi - \vec{\nabla}\phi^*\vec{\nabla}\phi - m^2\phi^*\phi) \right] \\ &= \int_{t=0} d\vec{x} \left[\partial_0\phi^*\partial_0\phi + \vec{\nabla}\phi^*\vec{\nabla}\phi + m^2\phi^*\phi) \right] \qquad (\geq 0) \\ &= 4\pi \int_0^\infty r^2 dr \, \frac{1}{r^2} \left[\left| \sum_n i n m_0 A_n e^{-\sqrt{m^2 - n^2 m_0^2} r} \right|^2 + \right. \\ &\qquad + \left| \sum_n A_n e^{-\sqrt{m^2-n^2m_0^2}r} \left(\sqrt{m^2 - n^2 m_0^2} + \frac{1}{r} \right) \right|^2 + \\ &\qquad \left. + m^2 \left| \sum_n A_n e^{-\sqrt{m^2-n^2m_0^2}r} \right|^2 \right] \end{aligned} \tag{3.10}$$

and the $1/r$ term in the second squared term makes the integral divergent. The integral converges if $\sum A_n = 0$ but the computation on another space-like hypersurface $t = t_0 \neq 0$ would be still divergent, which is an indication that the computation at $t = 0$ is meaningless, even if it can be accidentally convergent.

3.3 Solitons for the coupled SQED equations

The field v^μ may also be written in a simple generic form if we suppose that it obeys the spherically-symmetric soliton condition. The most general form compatible with the symmetries of the solution is given by:

$$v^\mu(x) = a(u)\lambda^\mu + b(u)p^\mu \tag{3.11}$$

$$v^2 = m_0^2(b^2 - a^2 u)\,. \tag{3.12}$$

The first term of v^μ does not contribute to the field strength tensor because if we set $A = \int_0^u a(s)ds$, then $a(u)\lambda^\mu = \partial^\mu(A(u)/2)$ which is a pure gauge term. And we will further demonstrate that this term must vanish. However, we will see in the next sections that for periodic solutions, this term is important.

We will also need to comply with the classical asymptotic conditions at infinity in space-like directions. One must therefore have A^μ decreasing as $1/r$ at infinity, and thus $b(u) \simeq C/\sqrt{u}$ when $u \to \infty$.

Then we can substitute v^μ and $z(x) = f(u)$ in the equations of motion eq. (2.11) and eq. (2.12):

$$(\Box + (m^2 - e^2 v^2))z = 0 \tag{3.13}$$

$$\Box(v_\nu) - \partial_\nu(\partial \cdot v) = -2e^2 z^2 v_\nu \,. \tag{3.14}$$

Using the parameterization of v^μ given in eq. (3.11) one gets:

$$\begin{aligned}
\partial_\alpha v_\beta &= 2\frac{d(a)}{du}\lambda_\alpha\lambda_\beta + 2\frac{d(b)}{du}\lambda_\alpha p_\beta + a\tau_{\alpha\beta} \\
F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu &= 2\frac{d(b)}{du}(\lambda \wedge p)^{\mu\nu} = -2m_0^2\frac{d(b)}{du}(x \wedge p)^{\mu\nu} \\
F_{\mu\lambda}F_\nu{}^\lambda &= 4m_0^4 \frac{d(b)}{du}^2 (x_\mu x_\nu m_0^2 - (p\cdot x)(p_\mu x_\nu + p_\nu x_\mu) + x^2 p_\mu p_\nu) \\
F^2 &= -8m_0^4\frac{d(b)}{du}^2 u \\
\Box v_\nu - \partial_\nu(\partial \cdot v) &= -4m_0^2 u\frac{d^2(b)}{du^2}p_\nu - 6m_0^2\frac{d(b)}{du}p_\nu \,.
\end{aligned} \tag{3.15}$$

Thus eq. (3.14) yields:

$$\begin{aligned}
-4m_0^2 u\frac{d^2(b)}{du^2}p_\nu - 6m_0^2\frac{d(b)}{du}p_\nu &= -2e^2 f^2(a\lambda_\nu + bp_\nu) \\
\Rightarrow \qquad a &= 0 \\
\text{and} \qquad 4u\frac{d^2(b)}{du^2} + 6\frac{d(b)}{du} &= \frac{2e^2}{m_0^2}f^2 b \\
b(u) = \frac{\tilde{b}(\sqrt{u})}{\sqrt{u}} \Rightarrow \qquad \tilde{b}'' &= \frac{2e^2}{m_0^2}f^2\tilde{b} \,.
\end{aligned} \tag{3.16}$$

Similarly, we will use the change of variable $f(u) = m_0 t(\sqrt{u})/\sqrt{u}$ in eq. (3.13) and in eq. (3.16). We finally obtain this system of coupled differential equations:

$$t''(y) - \left(\frac{m^2}{m_0^2} - e^2\frac{\tilde{b}(y)^2}{y^2}\right) t = 0 \tag{3.17}$$

$$\tilde{b}''(y) - 2e^2\frac{t(y)^2}{y^2}\tilde{b}(y) \,. = 0 \tag{3.18}$$

3.3.1 Normalization of the solutions

In this paragraph, we compute the conserved momentum of spherically symmetric solitons. We will show that the solutions cannot be normalized. Considering the energy-momentum tensor of eq. (2.14), we get for a soliton:

$$
\begin{aligned}
\vec{P} &= \vec{0} \\
F_{0\lambda}F_0{}^{\lambda} &= -4m_0^6\left(\frac{db}{du}\right)^2 r^2 \qquad (\text{rest frame}, t=0, r=|\vec{x}|) \\
v^0 &= m_0 b(u)\,; \qquad \partial_0 z = 0\,; \qquad (\partial_\mu z)^2 = -4m_0^2 u\left(\frac{df}{du}\right)^2 \\
T_{00} &= e^2 z^2(2m_0^2b^2 - m_0^2b^2 + m_0^2a^2u) + 4m_0^2u\left(\frac{df}{du}\right)^2 + \\
&\quad + m^2f^2 + 4m_0^6\left(\frac{db}{du}\right)^2 r^2 - 2m_0^4\left(\frac{db}{du}\right)^2 u \\
&= m_0^2e^2f^2b^2 + 4m_0^2u\left(\frac{df}{du}\right)^2 + m^2f^2 + 2m_0^6\left(\frac{db}{du}\right)^2 r^2 \\
&= e^2m_0^4\frac{t^2\tilde{b}^2}{y^4} + m^2\frac{t^2}{y^2} + 4m_0^6r^2\left(\frac{1}{2y}\frac{d}{dy}\left(\frac{t(y)}{y}\right)\right)^2 + 2m_0^6r^2\left(\frac{1}{2y}\frac{d}{dy}\left(\frac{\tilde{b}(y)}{y}\right)\right)^2 \\
&= e^2m_0^4\frac{t^2\tilde{b}^2}{y^4} + m^2\frac{t^2}{y^2} + m_0^4\left(\frac{d}{dy}\left(\frac{t(y)}{y}\right)\right)^2 + \frac{m_0^4}{2}\left(\frac{d}{dy}\left(\frac{\tilde{b}(y)}{y}\right)\right)^2 \\
P_0 &= 4\pi\int_0^\infty r^2\,dr\,T_{00} \\
P_0 &= \frac{4\pi}{m_0^3}\int_0^\infty y^2\,dy\left[e^2m_0^4\frac{t^2\tilde{b}^2}{y^4} + m^2\frac{t^2}{y^2} + m_0^4\left(\frac{d}{dy}\left(\frac{t(y)}{y}\right)\right)^2 + \right. \\
&\qquad\left. + \frac{m_0^4}{2}\left(\frac{d}{dy}\left(\frac{\tilde{b}(y)}{y}\right)\right)^2\right] \\
&= 4\pi m_0\int_0^\infty dy\left[e^2\frac{t^2\tilde{b}^2}{y^2} + \left(\frac{m}{m_0}\right)^2 t^2 + y^2\left(\frac{d}{dy}\left(\frac{t(y)}{y}\right)\right)^2 + \right. \\
&\qquad\left. + \frac{y^2}{2}\left(\frac{d}{dy}\left(\frac{\tilde{b}(y)}{y}\right)\right)^2\right]. \qquad (3.19)
\end{aligned}
$$

We have seen that $\tilde{b}$ must tend to a non-vanishing constant at infinity in space-like directions ($A^\mu \simeq 1/r$), but from eq. (3.18) we can conclude that $\tilde{b}$ is a convex function when $\tilde{b} > 0$ and the converse for the other sign. From the last term in eq. (3.19), we get that $\tilde{b}$ cannot tend to a non-vanishing value in $y = 0$ (otherwise the integral is divergent). Thus if $\tilde{b}$ vanish in $y = 0$, it cannot tend to a non-vanishing constant at infinity because it is a convex function if $\tilde{b} > 0$ or the converse if $\tilde{b} < 0$. The only possibility is $\tilde{b} = 0$, and we are then back to the free case, which we have previously rejected.

3.4 Is there some periodic solutions to the coupled equations?

Now we introduce a "time" variable $\tau = p \cdot x$ which is dimensionless and $y = \sqrt{u}$ like in the soliton case. We have:

$$\begin{aligned}
z(x) &= f(\tau, y) = \frac{t(\tau, y)}{y} \\
v^\mu &= a(\tau, y)\lambda^\mu + b(\tau, y)p^\mu \\
\Rightarrow \partial_\mu z &= p_\mu \partial_0 f + \frac{\lambda^\mu}{y}\partial_1 f \\
\Rightarrow \Box z &= m_0^2 \left(\partial_0^2 f - \frac{2}{y}\partial_1 f - \partial_1^2 f\right) = \frac{m_0^2}{y}\left(\partial_0^2 t - \partial_1^2 t\right) \qquad (3.20) \\
\Box v_\mu - \partial_\mu(\partial \cdot v) &= m_0^2 p_\mu \left[\rho \partial_0 \partial_1 a + 3\partial_0 a - \partial_1^2 b - \frac{2}{y}\partial_1 b\right] + m_0^2 \lambda_\mu \left[\partial_0^2 a - \frac{\partial_0 \partial_1 b}{y}\right].
\end{aligned}$$

From these basic calculations we get for the equations of motion:

$$\begin{aligned}
\partial_0^2 t - \partial_1^2 t + \left(\frac{m^2}{m_0^2} - e^2(b^2 - y^2 a^2)\right) t = 0& \\
\partial_0^2 a - \frac{\partial_0 \partial_1 b}{y} &= -2\frac{e^2 t^2}{m_0^2 y^2} a \\
y \partial_0 \partial_1 a + 3\partial_0 a - \partial_1^2 b - \frac{2}{y}\partial_1 b &= -2\frac{e^2 t^2}{m_0^2 y^2} b \,. \qquad (3.21)
\end{aligned}$$

These equations are much more complicated than in the case of solitons and the fundamental structure of the solutions, even periodic in time is not clear so far. We will restrict ourselves in this paragraph to a description of what is really different in this case and why we conjecture the existence of some normalized periodic solutions.

The conservation of the electromagnetic current leads to the emergence of a kind of pre-potential:

$$\begin{aligned}
\partial_\mu(z^2 v^\mu) &= 0 \\
\Leftrightarrow \partial_1(y t^2 a) &= \partial_0(t^2 b) \\
\Rightarrow y t^2 a &= \partial_0 \varphi(\tau, y) \\
\text{and} \qquad t^2 b &= \partial_1 \varphi(\tau, y)\,. \qquad (3.22)
\end{aligned}$$

Introducing this potential in the equations for the electromagnetic field we get:

$$\begin{aligned}
\partial_0 \left(\partial_0 a - \frac{\partial_1 b}{y}\right) &= -2\frac{e^2}{m_0^2}\partial_0 \left(\frac{\varphi}{y^3}\right) \\
\partial_1 \left(y^2 \partial_1 b\right) - \partial_1 \partial_0 (y^3 a) &= 2\frac{e^2}{m_0^2}\partial_1 \varphi\,. \qquad (3.23)
\end{aligned}$$

These equations can be partially integrated, and we obtain:

$$\partial_0 a - \frac{\partial_1 b}{y} = -2\frac{e^2}{m_0^2}\frac{\varphi}{y^3} + A(y) \tag{3.24}$$

$$y^2\partial_1 b - \partial_0(y^3 a) = 2\frac{e^2}{m_0^2}\varphi + B(\tau)\,. \tag{3.25}$$

The presence of these two functions A and B enlarges significantly the set of possibilities for the solutions. We therefore hope that some of these might be normalizable, as we shall discuss further.

3.4.1 Normalization of the time-dependent solutions

In order to normalize these periodic solutions, the computation of the conserved momentum gives for eq. 2.14:

$$\begin{aligned}
F_{\mu\nu} &= (p\wedge\lambda)_{\mu\nu}\left(\partial_0 a - \frac{\partial_1 b}{y}\right)\\
F_{\mu\alpha}F_\nu{}^\alpha &= -m_0^2\left(\partial_0 a - \frac{\partial_1 b}{y}\right)^2 (y^2 p_\mu p_\nu - \lambda_\mu\lambda_\nu)\\
\Rightarrow F_{0\alpha}F_\mu{}^\alpha &= -m_0^3 y^2\left(\partial_0 a - \frac{\partial_1 b}{y}\right)^2 p_\mu\\
F_{\beta\alpha}F^{\beta\alpha} &= -2m_0^4 y^2\left(\partial_0 a - \frac{\partial_1 b}{y}\right)^2\\
T_{0i} &= 2e^2 f^2 m_0 ab\lambda^i + 2m_0\partial_0 f\frac{\lambda^i}{y}\\
\Rightarrow P_i &= 0\\
T_{00} &= e^2 m_0^2 f^2(b^2 + a^2y^2) + m^2 f^2 + m_0^2\left((\partial_0 f)^2 + (\partial_1 f)^2\right) + \frac{y^2}{2}\left(\partial_0 a - \frac{\partial_1 b}{y}\right)^2\\
P_0 &= \frac{4\pi}{m_0^3}\int_0^\infty y^2 dy\, T_{00}\\
&= \frac{4\pi}{m_0}\int_0^\infty y^2 dy\left(e^2 f^2(b^2 + a^2y^2) + \frac{m^2}{m_0^2}f^2 + \left((\partial_0 f)^2 + (\partial_1 f)^2\right)\right.\\
&\qquad\left. + \frac{y^2}{2m_0^2}\left(\partial_0 a - \frac{\partial_1 b}{y}\right)^2\right)\,.
\end{aligned} \tag{3.26}$$

The last term in eq. (3.26) also appears in eq. (3.24), equation that was absent when we considered soliton solutions. In this equation, the function A is undetermined but if φ/y^3 is sufficiently singular at 0, the function A will certainly not compensate the singularity because it is time-independent, and φ is periodic. Thus, if A accidentally compensate φ/y^3 at $y = 0$ for $t = 0$, it may not be the case at a different time. As a consequence, φ must certainly vanish at $y = 0$ if one wants the integral to be convergent.

We still have in this case some dramatic constraints on the behaviour of the solutions at $y = 0$. However, what prevented us from finding normalized periodic solutions in the free case was the finite value of t at $y = 0$. In the free case, solutions are only composed of exponentially decreasing functions. Here we have another "mass" term in the equation of motion for t. If $b^2 - a^2y^2$ becomes large in the vicinity of the origin, one may obtain solutions that are spatially oscillating (and only near $y = 0$). Such a possibility allows to have a t function that vanishes at $y = 0$, while still featuring an exponentially decreasing behaviour at infinity. We expect soon to be able to confirm this conjecture by numerical simulations, before we can get more rigorous answers to this problem.

4. The non-abelian case

4.1 The standard equations of motion

In this case we consider a scalar field Φ lying in an N-dimensional vector space of representation of the Lie group $\mathcal{G}$ (a subgroup of $\mathrm{U}(N)$). The results presented in this section will not work for all the possible gauge groups, yet our method is valid for $\mathrm{U}(N)$ or $\mathrm{SU}(N)$. There are very few constraints that may be imposed on a generic gauge group. Probably the most important one is that there must exist a scalar product on the Lie algebra which is invariant under an inner automorphism. One can then demonstrate that the solvable part (in the Levy decomposition of the group) must be abelian. Thus if we also constrain the group to be compact, it is relevant to consider gauge groups as being a sum of $\mathrm{U}(1)$ terms, plus any semi-simple part like $\mathrm{SU}(N)$. Since the $\mathrm{U}(1)$ case has been previously solved, we focus here on $\mathrm{SU}(N)$ groups. Actually, we will see that our formalism works if the orbit of any vector Φ under the gauge group is C^N, which is the case for $\mathrm{SU}(N)$.

We will denote by $i\mathcal{A}$ the real Lie algebra, such that the matrices lying in $\mathcal{A}$ are hermitean. If ρ stands for the representation of the Lie algebra, we can endow the algebra with the following scalar product for the computation of the Yang-Mills part of the lagrangian: $(A, B)_\rho = \mathrm{Tr}[\rho(A)^\dagger \rho(B)]$. The scalar product generally used with a semi-simple group is the Killing form applied to the field strength tensor. Since this scalar product is proportional to any scalar product of the form $(A, B)_\rho$ (the coefficient being the Dynkin index of ρ), we will simply use this scalar product $(A, B) = \mathrm{Tr}[A^\dagger B] = \mathrm{Tr}[AB]$. In the following, we give the lagrangian and the corresponding equations of motion, using Φ and W_μ as variables.

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_{YM} \\
\mathcal{L}_0 &= (D_\mu\Phi)^\dagger D^\mu\Phi - m^2\Phi^\dagger\Phi \\
&= \partial_\mu\Phi^\dagger\partial^\mu\Phi - ig\Phi^\dagger W_\mu\partial^\mu\Phi + ig\partial_\mu\Phi^\dagger W^\mu\Phi + \\
&\quad + g^2\Phi^\dagger W_\mu W^\mu\Phi - m^2\Phi^\dagger\Phi
\end{aligned}$$

$$
\begin{aligned}
\mathcal{L}_{YM} &= -\frac{1}{4}(F_{\mu\nu}, F^{\mu\nu}) \\
&= -\frac{1}{4}\Big\{ \operatorname{Tr}\left[(\partial_\mu W_\nu - \partial_\nu W_\mu)(\partial^\mu W^\nu - \partial^\nu W^\mu)\right] - \\
&\qquad - g^2 \operatorname{Tr}\left[[W_\mu, W_\nu][W^\mu, W^\nu]\right] + \\
&\qquad + 2ig \operatorname{Tr}\left[(\partial_\mu W_\nu - \partial_\nu W_\mu)[W^\mu, W^\nu]\right] \Big\} \\
\frac{\partial \mathcal{L}}{\partial(\partial_\mu W_\nu)}(\Omega_\nu) &= -\ \operatorname{Tr}[\Omega_\nu G^{\mu\nu}] \\
\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu W_\nu)}(\Omega_\nu) - & \\
-\frac{\partial \mathcal{L}}{\partial(W_\nu)}(\Omega_\nu) &= -\ \operatorname{Tr}[\Omega_\nu \partial_\mu G^{\mu\nu}] - \\
&\quad - \left(-ig\Phi^\dagger \Omega_\mu \partial^\mu \Phi + ig\partial_\mu \Phi^\dagger \Omega^\mu \Phi + g^2 \Phi^\dagger \{\Omega_\mu, W^\mu\}\Phi\right) + \\
&\quad + ig \operatorname{Tr}\left[[W_\mu, \Omega_\nu] G^{\mu\nu}\right] \\
&= -\ \operatorname{Tr}[\Omega_\nu \partial_\mu G^{\mu\nu}] + ig \operatorname{Tr}\left[\Omega_\nu \left(D^\nu \Phi \Phi^\dagger - \Phi (D^\nu \Phi)^\dagger\right)\right] \\
&\quad + ig \operatorname{Tr}\left[\Omega_\nu [G^{\mu\nu}, W_\mu]\right] \\
\Rightarrow 0 &= \operatorname{Tr}\left[\Omega_\nu \left(-\mathcal{D}_\mu G^{\mu\nu} + ig(D^\nu \Phi \Phi^\dagger - \Phi (D^\nu \Phi)^\dagger)\right)\right] \qquad (\forall\ \Omega_\nu\ \in\ \mathcal{A}) \\
\mathcal{D}^\mu (G_{\mu\nu}) &= \Pi_{\mathcal{A}}\left[ig\left(D_\nu \Phi \Phi^\dagger - \Phi (D_\nu \Phi)^\dagger\right)\right] = \Pi_{\mathcal{A}}(J_\nu) \\
J_\mu &= ig\left(\partial_\mu \Phi \Phi^\dagger - \Phi \partial_\mu \Phi^\dagger + ig\{W_\mu, \Phi\Phi^\dagger\}\right) \\
\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^\dagger)} - \frac{\partial \mathcal{L}}{\partial(\Phi^\dagger)} &= 0 \\
\Rightarrow 0 &= (D_\mu D^\mu + m^2)\Phi \\
&= (\Box + m^2)\Phi + 2igW_\alpha \partial^\alpha \Phi + ig(\partial_\alpha W^\alpha)\Phi - g^2 W_\mu W^\mu \Phi\,. \qquad (4.1)
\end{aligned}
$$

We shall note that in eq. (4.1), the fact that the equation is only valid for $\Omega_\nu \ \in\ \mathcal{A}$ is very important. It comes from the fact that in the variationnal principle leading to the Euler-Lagrange equations, the variation of the gauge field (Ω_ν) must lie in the Lie algebra also. If this equation were valid for any matrix Ω_ν, then we would have $\mathcal{D}^\alpha(G_{\alpha\beta})$ (which is in $\mathcal{A}$) equal to $ig\left(D_\beta \Phi \Phi^\dagger - \Phi (D_\beta \Phi)^\dagger\right)$, which is not necessarily in $\mathcal{A}$, and this is why there is this projection operator on the Lie algebra $\Pi_{\mathcal{A}}$ in eq. 4.1. For su(N) algebras, this projection is simply $M \mapsto M - \operatorname{Tr}(M)\frac{I}{N}$, where I stands for the identity matrix. eq. (4.1) and eq. (4.1) are the equations of motion respectively for the gauge fields and for the scalar fields, which can be related to the abelian equations of eq. (2.4) and eq. (2.3). The non-abelian equivalent of the current is now extracted from eq. (4.1) and is given by the matrix: $J_\nu = ig(D_\nu \Phi \Phi^\dagger - \Phi (D_\nu \Phi)^\dagger)$.

Contrary to the abelian case, this current is not gauge invariant anymaore but rather gauge covariant, that is $J_\nu = U J'_\nu U^{-1}$ under a gauge transformation.

We now operate as in the previous section, and observe that if we compute $(4.1)\Phi^\dagger - \Phi(4.1)^\dagger$, we get:

$$\begin{aligned} 0 &= (D_\mu D^\mu \Phi)\Phi^\dagger - \Phi(D_\mu D^\mu \Phi)^\dagger \Phi \\ &= \mathcal{D}_\mu \left((D^\mu \Phi)\Phi^\dagger - \Phi(D^\mu\Phi)^\dagger\right) \\ \Rightarrow 0 &= \mathcal{D}_\mu \left(J^\mu\right). \end{aligned} \tag{4.2}$$

If one projects this equation on the Lie algebra, the resulting equation is redundant with eq. (4.1) on which we apply the operator $\mathcal{D}^\nu$. Like in the abelian case, we find a redundancy, but it is important to note at this stage that eq. (4.2) is a stronger condition than if we just applied $\mathcal{D}^\nu$ on eq. (4.1). It seems that we missed some degrees of freedom in eq. (4.1). The fundamental structure of the gauge group is responsible for this fact. For instance, in the case of a $u(N)$ algebra, $\Pi_{\mathcal{A}}(M) = M$ if M is hermitean, and all the "degrees of freedom" of J_μ are concerned with this redundancy between the equation for the matter and the equation for the gauge field.

We therefore have too much information in the set of equations of the matter field and one should replace eq. (4.1) by $(4.1)\Phi^\dagger + \Phi(4.1)^\dagger$, i.e.:

$$\begin{aligned} 0 &= (D_\mu D^\mu \Phi)\Phi^\dagger + \Phi(D_\mu D^\mu\Phi)^\dagger + 2m^2\Phi\Phi^\dagger \\ &= \mathcal{D}_\mu\left((D^\mu\Phi)\Phi^\dagger + \Phi(D^\mu\Phi)^\dagger\right) - 2D_\mu\Phi(D^\mu\Phi)^\dagger + 2m^2\Phi\Phi^\dagger \\ &= \left(\mathcal{D}_\mu\mathcal{D}^\mu(\Phi\Phi^\dagger) - 2(D_\mu\Phi)(D^\mu\Phi)^\dagger + 2m^2\Phi\Phi^\dagger\right). \end{aligned} \tag{4.3}$$

4.2 The gauge invariant variables

The procedure used to obtain eq. (2.8) consists in eliminating the two first terms of $J_\mu/ie = \varphi^*\partial_\mu\varphi - \partial_\mu\varphi^*\varphi + 2ie\rho A_\mu$ in order to extract the gauge field. We have $J_\mu/ie\rho = 2ieA_\mu + \partial_\mu\Lambda$ and the pure gauge term disappears in $F_{\mu\nu}$. But in our case we have a matrix and this procedure does not work. However, the extraction of A_μ can be seen in another way. In the abelian case, we could also have taken a unitary gauge, that is to say a gauge in which φ is real. This automatically eliminates the desired terms. We may proceed here in a similar way. The essential hypothesis is that any two scalar fields Φ and Ψ_0 can be related by an element of the gauge group. It is the case for $\mathrm{U}(N)$ or $\mathrm{SU}(N)$. Thus, the central point of the method is to choose a constant unitary vector Ψ_0, and therefore one can find U in the gauge group such that:

$$\begin{aligned} \Phi &= zU\Psi_0 \\ z &= \sqrt{\Phi^\dagger\Phi} = \sqrt{\rho}. \end{aligned} \tag{4.4}$$

A consequence is that if ${W'}_\mu$ is the gauge field in the "unitary" gauge obtained by the matrix U we have from eq. 4.1:

$$\begin{aligned} {J'}_\mu = U^{-1}J_\mu U &= ig\left(z\partial_\mu(z)\Psi_0\Psi_0^\dagger - \Psi_0\Psi_0^\dagger z\partial_\mu(z) + ig\rho\{{W'}_\mu, \Psi_0\Psi_0^\dagger\}\right) \\ &= (ig)^2\rho\{{W'}_\mu, \Psi_0\Psi_0^\dagger\}. \end{aligned} \tag{4.5}$$

However, it is in general impossible to reconstruct the entire gauge field ${W'}_\mu$ from this equation, except for the SU(2) case because of the relation $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$ (and this anticommutator has no residue lying in the Lie algebra). Using this property, the traceless part of ${J'}_\mu$ gives $(ig)^2\rho {W'}_\mu$. Since it works only for SU(2), we need to find a way to get the missing degrees of freedom of the gauge field. The method consists in constructing an orthonormal basis of C^N, starting from Ψ_0: $(\Psi_0, \Psi_1, \ldots, \Psi_{N-1})$, which does not depend on space-time coordinates. If we set $\Phi_k = zU\Psi_k$ $(k \geq 1)$, then $(\rho^{-1}\Phi, \rho^{-1}\Phi_1, \ldots, \rho^{-1}\Phi_{N-1})$ forms also an orthonormal basis. A gauge transformation will naturally apply also to these new scalar fields, and we consider the gauge invariant variables:

$$J_{mn\mu} = ig\left(\Phi_m^\dagger D_\mu \Phi_n - (D_\mu\Phi_m)^\dagger\Phi_n\right). \tag{4.6}$$

The simple reason why we do not consider some other gauge invariant variables, by taking the sum of the two terms above instead of their difference is that $\Phi_m^\dagger D_\mu \Phi_n + (D_\mu\Phi_m)^\dagger\Phi_n = \partial_\mu(\Phi_m^\dagger\Phi_n) = \partial_\mu(\rho\delta_{m,n})$ and thus they can be expressed using the gauge invariant variable $z = \sqrt{\rho}$. In the unitary gauge, these gauge invariant variables allow to reconstruct the gauge field completely:

$$\begin{aligned}
J_{mn\mu} &= 2(ig)^2\rho\Psi_m^\dagger {W'}_\mu \Psi_n\,, \qquad (J_{nm\mu} = J^*_{mn\mu}) \\
v_{mn\mu} &= \frac{-1}{2g^2\rho} J_{mn\mu} \\
\Rightarrow {W'}_\mu &= \sum_{m,n} v_{mn\mu}\Psi_n\Psi_m^\dagger\,.
\end{aligned} \tag{4.7}$$

The equations of motion for the gauge field in eq. (4.1) can then be rewritten in the unitary gauge (note that $\Pi_{\mathcal{A}}(U^{-1}JU) = U^{-1}\Pi_{\mathcal{A}}(J)U$)

$${G'}_{\mu\nu} = \sum_{m,n}\left(\partial_\mu\left(v_{mn\nu}\right) - \partial_\nu\left(v_{mn\mu}\right) + ig\sum_k (v_{mk\mu}v_{kn\nu} - v_{mk\nu}v_{kn\mu})\right)\Psi_n\Psi_m^\dagger \tag{4.8}$$

$$\mathcal{D}'^\mu({G'}_{\mu\nu}) = \partial^\mu {G'}_{\mu\nu} + ig[{W'}_\mu, {G'}_{\mu\nu}] \tag{4.9}$$

$$\begin{aligned}
&= \sum_{m,n}\left(\Box\left(v_{mn\nu}\right) - \partial_\nu\left(\partial\cdot v_{mn}\right) + ig\sum_k \partial^\mu (v_{mk\mu}v_{kn\nu} - v_{mk\nu}v_{kn\mu})\right)\Psi_n\Psi_m^\dagger + \\
&\quad + ig\sum_{m,n}\Psi_n\Psi_m^\dagger\left[\sum_l {v_{ml}}^\mu\left(\partial_\mu\left(v_{ln\nu}\right) - \partial_\nu\left(v_{ln\mu}\right) + \right.\right. \\
&\qquad\qquad \left. + ig\sum_k (v_{lk\mu}v_{kn\nu} - v_{lk\nu}v_{kn\mu})\right) - \\
&\qquad\qquad - \left(\partial_\mu\left(v_{ml\nu}\right) - \partial_\nu\left(v_{ml\mu}\right) + \right. \\
&\qquad\qquad \left.\left. + ig\sum_k (v_{mk\mu}v_{kl\nu} - v_{mk\nu}v_{kl\mu})\right){v_{ln}}^\mu\right]
\end{aligned} \tag{4.10}$$

$$= -g^2 z^2 \sum_m \Pi_{\mathcal{A}} \left(v_{0m\mu} \Psi_m \Psi_0^\dagger + v_{m0\mu} \Psi_0 \Psi_m^\dagger \right) \tag{4.11}$$

$$= -g^2 z^2 \sum_m \left(v_{0m\mu} \Psi_m \Psi_0^\dagger + v_{m0\mu} \Psi_0 \Psi_m^\dagger - 2 \frac{v_{00\mu}}{N} \Psi_m \Psi_m^\dagger \right). \tag{4.12}$$

The last equality is only valid for SU(N). One must adapt this formula for another gauge group. Projecting these equations on the basis of matrices $\Psi_m \Psi_n^\dagger$ leads to a large set of N^2 equations in which only gauge invariant variables are present. In the SU(N) case we can also separate these equations into four different classes depending on the indices m and n, because of the specific form of the current matrix projected on the Lie algebra. The four cases correspond to the diagonal case with indices in the form (m, m) $(m > 0)$, the case with indices in the form $(0, m)$ or $(m, 0)$ $(m > 0)$, and finally the case where $m = n = 0$. The projection on these different cases can be easily done and we will not present them here. It is clear that the gauge fields ${W'}_\mu$ expressed in the basis of the Ψ_k's is nothing but the matrix composed of the gauge invariant coefficients $(v_{m,n\mu})$. Of course, these coefficients depend on the constant basis we choose, but physical solutions must be independent of this choice. It remains to demonstrate that these equations of motion can be re-expressed using only variables that are also independent from the constant basis chosen: we can consider some objects of the form $\mathrm{Tr}[({W'}_\mu)^n]$, or equivalently the characteristic polynomial of ${W'}_\mu$. We expect to have new results in the near future.

We may conclude this last section with the equation of motion for the matter fields. The simplest way is to look at the lagrangian and to use the following equality:

$$\frac{1}{-g^2} \sum_{m,n} J_{mn\mu} J_{nm}{}^\mu = N \partial_\mu \rho \partial^\mu \rho - 4\rho (D_\mu \Phi)^\dagger (D^\mu \Phi). \tag{4.13}$$

The matter part of the lagrangian can then be written:

$$\mathcal{L}_0 = N \partial_\mu z \partial^\mu z - m^2 z^2 + g^2 z^2 \sum_{m,n} v_{mn\mu} v_{nm}{}^\mu. \tag{4.14}$$

And the equation of motion for the scalar field is finally:

$$\left(\Box + \frac{m^2}{N} \right) z = \frac{g^2}{N} z \sum_{m,n} v_{mn\mu} v_{nm}{}^\mu. \tag{4.15}$$

5. Conclusion

In this paper, we give a certain number of results which are really encouraging for the purpose of reformulating gauge theories using only gauge-invariant variables. Within the prospects of this work, a short-term project would naturally be to find an equivalent formulation when fermions are involved. Then, the quantization of the theory has to be constructed. Within this subtopic, it would be interesting to revisit the general formalism of quantization in QFT. An equation like $dA/dt = i[H, A]$ is a very old non-relativistic formula which is surprisingly still used in textbooks about relativistic quantum field theory. Instead of the hamiltonian, one would naturally consider an operator of the form $\int_\Sigma d\sigma^\mu T_{\mu\nu}$ in order to quantize a theory. This has not been done yet and one of the possible reasons is that there is no unique expression for the energy-momentum tensor $T_{\mu\nu}$. There are some current research activities on this topic [14], in order to find the "best" criteria to define uniquely $T_{\mu\nu}$. So far, it seems that the Belinfante tensor is a good candidate, since it is gauge-invariant. Therefore it can be naturally inserted in the formalism presented in this paper. Finally, in the long-term we hope to be able to compute some scattering cross sections using directly gauge invariant variables, and also to provide a revised version of Quantum Field Theory which would apply to unstable particles and more generally, to physical systems that evolve on a "long-time" scale, (CP violation, neutrino oscillations,...) as mentionned in the introduction.

A. Review of basic cohomological formulas

As noted in this paper, one of the main problems regarding gauge independence is to have a method to find the set of gauge fields with a given Field-Strength tensor $F^{\mu\nu}$. We will separate the abelian case from the non-abelian one, because the curvature tensor depends linearly on the gauge field in the abelian case, quadratically in the latter case. Linearity is lost in the non-abelian case, which renders the problem much more complicated.

The problem can be summarized as follows: if one has a specific tensor F of rank n, we look for another tensor of rank $n-1$ such that $F = dA$ where d represents the exterior derivative. The tensor F must obey $dF = 0$ because of the property $d^2 = 0$. So we want to find A from a given F, assumed that F is a closed form (i.e. $dF = 0$). Given a solution A, one can find another solution A' by adding to A any term of the form $d\Lambda$, again because $d^2 = 0$. Therefore, we will say that two tensors of rank $n-1$ are co-homologous if there exists Λ such that $A - A' = d\Lambda$. It is an equivalence relation and the equivalence classes are called cohomology classes (for the de-Rahm cohomology, and we will further explain why it is important to make this distinction when the non-abelian case is involved).

B. Abelian gauge fields

Let $M = R^4$ be the Minkowsky space-time, and consider $X^\mu(u, x)$ an application from $[0, 1] \times M$ into M such that:

$$\forall x \in M, X^\mu(0, x) = x_0^\mu \tag{B.1}$$

$$\forall x \in M, X^\mu(1, x) = x^\mu \,. \tag{B.2}$$

We also assume that X^μ is infinitely smooth. It is then called a "contraction". The reader will recover the standard Poincaré formula by taking $X^\mu(u,x) = ux^\mu$. Suppose $A^\mu(x)$ is a vector field with vanishing curvature, then if we define $V(x)$ as follows:

$$V(x) = \int_0^1 du \frac{\partial X^\mu}{\partial u} A_\mu(X(u,x)) \,. \tag{B.3}$$

Then

$$\partial_\mu V = A_\mu(x) - \int_0^1 du \frac{\partial X^\alpha}{\partial u} \frac{\partial X^\beta}{\partial x^\mu} F_{\alpha\beta}(X) \,. \tag{B.4}$$

Therefore, if the curvature of A vanishes, $V(x)$ is a possible solution for the potential. Also, if one replaces explicitly A^μ by $\partial^\mu V'$ in eq. B.3, one gets $V'(x) - V'(x_0)$, and not $V'(x)$. $V(x)$ is therefore not a "fixed point solution" of an integral equation, but can be defined as the solution for which $V(x_0) = 0$. The rest in the expression of $\partial_\mu V$ vanishes explicitly for a vanishing curvature, but when the curvature is not 0, this formula provides us with an explicit expression for A^μ as a function of $F^{\mu\nu}$ up to a gauge transformation by $\partial^\mu V$. Thus we have already the next step, and if we consider a given field-strength tensor $F^{\mu\nu}$, we can define the following vector field:

$$A^\mu(x) = \int_0^1 du \frac{\partial X^\alpha}{\partial u} \frac{\partial X^\beta}{\partial x^\mu} F_{\alpha\beta}(X(u,x)) \,. \tag{B.5}$$

Then, with this definition we have:

$$\partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}(x) - \int_0^1 du \frac{\partial X^\alpha}{\partial u} \frac{\partial X^\beta}{\partial x^\mu} \frac{\partial X^\gamma}{\partial x^\nu} (\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta}) \,. \tag{B.6}$$

The last term vanishes if $dF = 0$, and we recognize here the homogeneous Maxwell equations. In this case, the expression we have chosen for A^μ is a possible gauge field, and this formula is of course very important because it allows us to "parameterize" the orbits of gauge fields. It is possible to go on with this scheme, and for a given 3-form $\omega_{\alpha\beta\gamma}$ we can define $F^{\mu\nu}$ using:

$$F_{\mu\nu} = \int_0^1 \frac{\partial X^\alpha}{\partial u} \frac{\partial X^\beta}{\partial x^\mu} \frac{\partial X^\gamma}{\partial x^\nu} \omega_{\alpha\beta\gamma}(X(u,x)) \tag{B.7}$$

and when $d\omega = 0$, we have $\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = \omega_{\alpha\beta\gamma}(x)$, and so on (but there is actually only one next step because we have assumed here that we are in four space-time dimensions and any four form is proportional to the Levi-Civita pseudo-tensor).

To summarize, given an n form F such that $dF = 0$, we have been able to exhibit a $n-1$ form A such that $F = dA$. This element A can be interpreted as an element of an equivalent class of cohomology with a given curvature. In other words, we have "computed" the cohomology. Expressed this way, it looks simple but hides the real difficulties, which are of a topological nature. In all these calculations, we have assumed the existence of X^μ, which imposes some constraints on the topology of the four dimensional space-time. If the whole Minkowsky space is taken under consideration, no topological problem occurs, and more generally, this is true if we consider a simply connected space. Then, one can find X^μ and proceed to the previous calculations.

C. Conventions for the non-abelian case

iA and iB are supposed to lie in the real Lie algebra corresponding to the Lie Group $\mathcal{G}$, which is a subgroup of $\mathrm{U}(N)$ here. Therefore A and B are hermitean. We set $A = UA'U^{-1}$. X and Y are vectors lying in the same representation as the matter field Φ.

$$
\begin{aligned}
\Phi &= U\Phi' = e^{iT}\Phi' \qquad (T \text{ small}) \\
W_\mu &= U{W'}_\mu U^{-1} + \frac{i}{g}\partial_\mu(U)U^{-1} \\
{W'}_\mu &= U^{-1}W_\mu U - \frac{i}{g}U^{-1}\partial_\mu(U) \\
\delta W_\mu = {W'}_\mu - W_\mu &= -\frac{i}{g}{\mathcal{D}'}_\mu(U)U^{-1} = -\frac{i}{g}U^{-1}\mathcal{D}_\mu(U) \\
U{W'}_\nu U^{-1} - {W'}_\nu + & \\
\frac{i}{g}(\partial_\nu U)U^{-1} &= e^{iT}{W'}_\nu e^{-iT} - {W'}_\nu + \frac{i}{g}(\partial_\nu e^{iT})e^{-iT} \\
&\simeq [iT, {W'}_\nu] - \frac{1}{g}(\partial_\nu T) = -\frac{1}{g}\mathcal{D}_\nu(A) \\
D_\mu\Phi &= (\partial_\mu + igW_\mu)\Phi \Rightarrow D_\mu\Phi = D_\mu(U\Phi') = U{D'}_\mu\Phi' \\
\mathcal{D}_\mu(A) &= \partial_\mu A + ig[W_\mu, A] \Rightarrow \mathcal{D}_\mu(UA'U^{-1}) = U({\mathcal{D}'}_\mu A')U^{-1} \\
\mathcal{D}_\mu(AB) &= \mathcal{D}_\mu(A)B + A\mathcal{D}_\mu(B) \\
\mathcal{D}_\mu(XY^\dagger) &= D_\mu(X)Y^\dagger + X(D_\mu(Y))^\dagger \\
D_\mu(AX) &= \mathcal{D}_\mu(A)X + AD_\mu(X) \\
[D_\mu, D_\nu]\Phi &= ig(\partial_\mu W_\nu - \partial_\nu W_\mu + ig[W_\mu, W_\nu])\Phi = igG_{\mu\nu}\Phi \\
G_{\mu\nu} &= U{G'}_{\mu\nu}U^{-1} \\
[\mathcal{D}_\alpha, \mathcal{D}_\beta](A) &= ig[G_{\alpha\beta}, A] \\
0 &= [D_\nu, [D_\rho, D_\sigma]]\Phi + [D_\rho, [D_\sigma, D_\nu]]\Phi + [D_\sigma, [D_\nu, D_\rho]]\Phi \\
\Leftrightarrow 0 &= \varepsilon^{\mu\nu\rho\sigma}[D_\nu, G_{\rho\sigma}](\Phi)\,, \qquad (\forall\,\Phi) \\
\Leftrightarrow 0 &= \mathcal{D}_\nu(\tilde{G}^{\mu\nu})\,, \qquad (\text{Bianchi})\,.
\end{aligned}
\tag{C.1}
$$

For $\mathrm{SU}(N)$ gauge groups, it may be useful to use the relation:

$$
\Phi\Phi^\dagger = \Phi^\dagger\Phi\frac{1}{N}I + A_\phi\,, \tag{C.2}
$$

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where I stands for the identity matrix in N dimensions, A_Φ lies therefore in the Lie algebra su(N), and we will conveniently denote by $\rho_\Phi = \Phi^\dagger\Phi$ the probability density of Φ.

D. Non abelian case and the Path Ordered Exponential

If A is an operator valued function of the real variable λ, a solution to the differential equation $f'(\lambda) = A(\lambda)f(\lambda)$ is given by (see [15]):

$$\begin{aligned} f(x) &= \Big[1 + \int_0^x d\lambda\, A(\lambda) + \int_0^x d\lambda_1\, A(\lambda_1) \int_0^{\lambda_1} d\lambda_2\, A(\lambda_2) + \cdots \\ &\quad + \int_0^x d\lambda_1\, A(\lambda_1) \times \cdots \times \int_0^{\lambda_{n-1}} d\lambda_n\, A(\lambda_n)\Big]\, f(0) \\ &= \underset{\rightarrow}{e}^{\int_0^x d\lambda\, A(\lambda)}\, f(0) = \lim_{ds\to 0} \prod_{k=n}^{1} e^{A(s_k)ds} f(0) \qquad \text{with } s_k = \frac{k\times x}{n} \end{aligned} \tag{D.1}$$

$$\underset{\rightarrow}{e}^{\int_0^x d\lambda\, A(\lambda)} = \underset{\rightarrow}{e}^{\int_y^x d\lambda\, A(\lambda)}\, \underset{\rightarrow}{e}^{\int_0^y d\lambda\, A(\lambda)}$$

$$\frac{d}{ds}\, \underset{\rightarrow}{e}^{\int_0^s F(v)dv} = F(s)\, \underset{\rightarrow}{e}^{\int_0^s F(v)dv} \qquad \frac{d}{ds}\, \underset{\rightarrow}{e}^{\int_s^1 F(v)dv} = -\, \underset{\rightarrow}{e}^{\int_0^s F(v)dv}\, F(s)\,. \tag{D.2}$$

Note that the product in eq. (D.1) is done "from right to left". In the following, we list a few properties of the path order exponential:

$$\begin{aligned} \left(\underset{\rightarrow}{e}^{\int A}\right)^{-1} &= \underset{\leftarrow}{e}^{-\int A} \\ &= 1 - \int_0^1 A(u)du + \int_0^1 du_1 \int_0^{u_1} du_2 A(u_2)A(u_1) + \cdots \\ \underset{\rightarrow}{e}^{\int_a^x A'(s)A^{-1}(s)ds} &= A(x)A^{-1}(a) \\ P(x) = \underset{\rightarrow}{e}^{\int_a^x A(s)ds} &\Rightarrow \underset{\rightarrow}{e}^{\int_a^x A(s)+B(s)ds} \\ &= P(x)\, \underset{\rightarrow}{e}^{\int_a^x P^{-1}(s)B(s)P(s)ds} \\ A(x)\, \underset{\rightarrow}{e}^{\int_a^x B(s)ds} A^{-1}(a) &= \underset{\rightarrow}{e}^{\int_a^x \left(A'(s)A^{-1}(s)+A(s)B(s)A^{-1}(s)\right)ds} \\ \frac{\partial}{\partial\lambda}\, \underset{\rightarrow}{e}^{\int_a^b A(u,\lambda)du} &= \int_a^b ds\, \underset{\rightarrow}{e}^{\int_s^b A(u,\lambda)du}\, \frac{\partial A(s,\lambda)}{\partial\lambda}\, \underset{\rightarrow}{e}^{\int_a^s A(u,\lambda)du}\,. \end{aligned} \tag{D.3}$$

The last formula can be demonstrated easily if one uses the product form of the ordered exponential (eq. (D.1))

D.1 Introduction of a space-time contraction

If we now consider a contraction $X_\mu(u,x)$ where $X_\mu(0,x) = x_0$ and $X_\mu(1,x) = x$ (see eq. (B.2)), we obtain the following definition:

$$
\begin{aligned}
F(u,x) &= \frac{\partial X_\mu}{\partial u}(u,x) A^\mu(X_\mu(u,x)) \\
\partial_\mu X_\alpha A^\alpha(X_\mu(u,x))|_{u=1} &= A_\mu(x) \\
f(x) &= \underset{\rightarrow}{e}^{ig\int_\gamma dl^\mu A_\mu}\ (x) \\
&= 1 + (ig)\int_0^1 du\, F(u,x) + \\
&\quad +(ig)^2 \int_0^1 du_1\, F(u_1,x) \int_0^{u_1} du_2 F(u_2,x) + \ldots \\
&\quad +(ig)^n \int_0^1 du_1\, F(u_1,x)\cdot\ldots\cdot \int_0^{u_{n-1}} du_n\, F(u_n,x) + \cdots \\
&= \sum_k (ig)^k \int_{[0;1]^k} du_1 \ldots du_k \theta(u_1,\ldots,u_k) F(u_1,x)\ldots F(u_k,x) \\
&= \exp\left(ig\int_0^1 du\, F(u,x)\right) \qquad (if\ [A(x),A(x')] = 0) \\
\theta(u_1,\ldots,u_k) &= \begin{cases} 1 & \text{if } u_1 \geq u_2 \ldots \geq u_k, \\ 0 & \text{if not} \end{cases} \\
&= H(u_1-u_2)H(u_2-u_3)\ldots H(u_{k-1}-u_k)\,.
\end{aligned}
\tag{D.4}
$$

Each term in the sum can be obtain by the following recursion:

$$
\begin{aligned}
J_0(a,b,x) &= 1 \\
J_n(a,b,x) &= \int_a^b ds\, F(s,x) J_{n-1}(a,s) \\
J_n(a,a,x) &= 0 \quad \forall\ n,x \\
J_n(a,b,x) &= \int_a^b ds\, \partial_s X_\mu A^\mu(X_\mu(s,x)) J_{n-1}(a,s)\,.
\end{aligned}
\tag{D.5}
$$

Let Φ be a solution (if it exists) to the system of PDE $\partial_\mu\Phi = -igW_\mu(x)\Phi$, then:

$$
\begin{aligned}
\frac{\partial X^\mu}{\partial u}\partial_\mu\Phi &= -ig\frac{\partial X^\mu}{\partial u}W_\mu(x)\Phi \\
\frac{d}{du}\Phi(X(u,x)) &= F(u,x)\Phi(X(u,x)) \\
\Rightarrow \Phi(X(u,x)) &= \underset{\rightarrow}{e}^{\int_0^u dv\, F(v,x)}\ \Phi_0 \\
\Rightarrow \Phi(x) = \Phi(X(1,x)) &= \underset{\rightarrow}{e}^{\int_0^1 dv\, F(v,x)}\ \Phi_0 \\
&= \underset{\rightarrow}{e}^{-ig\int_0^1 du\, \frac{\partial X^\mu}{\partial u} W_\mu(X(u,x))}\ \Phi_0\,.
\end{aligned}
\tag{D.6}
$$

If Φ is a square matrix and $\Phi_0 = I$, then Φ is invertible because $\det(\Phi) = e^{-ig\int \mathrm{Tr}\, F} \neq 0$, thus $W_\mu = \frac{i}{g}\partial_\mu \Phi \Phi^{-1}$ which is a right invariant form, the curvature of which vanishes. It is not surprising to get such a constraint. Already in the abelian case, if $\phi = e^{-ig\int A}$ then $\partial_\mu \phi = -ig\left(A_\mu + \int \partial_u X^\alpha \partial_\mu X^\beta F_{\alpha\beta}\right)\phi$ (see eq. (B.4)) and we explicitly show the presence of a curvature term as an obstacle to solve the system of differential equations. To obtain a similar formula in the non-abelian case, let us take the partial derivatives of eq. D.6. We get:

$$\begin{aligned}
\frac{i}{g}\partial_\mu \Phi &= \int_0^1 ds\ \underset{\rightarrow}{e}^{-ig\int_s^1 W_x}\ \partial_\mu\left(\frac{\partial X^\nu}{\partial s} W_\nu(X(s,x))\right)\ \underset{\rightarrow}{e}^{-ig\int_0^s W_x} \\
&= \int_0^1 ds\ \underset{\rightarrow}{e}^{-ig\int_s^1 W_x} \left(\partial_\mu \partial_s X^\nu W_\nu(X(s,x)) + \partial_s X^\nu \partial_\mu X^\rho \partial_\rho W_\nu(X(s,x))\right)\ \underset{\rightarrow}{e}^{-ig\int_0^s W_x} \\
&= \int_0^1 ds\ \underset{\rightarrow}{e}^{-ig\int_s^1 W_x}\ \{\partial_\mu \partial_s X^\nu W_\nu(X) + \partial_u X^\nu \partial_\mu X^\rho G_{\rho\nu}(X) + \\
&\quad + \partial_s X^\nu \partial_\mu X^\rho (\partial_\nu W_\rho(X) - ig[W_\rho, W_\nu])\}\ \underset{\rightarrow}{e}^{-ig\int_0^s W_x} \\
&= \int_0^1 ds\ \underset{\rightarrow}{e}^{-ig\int_s^1 W_x}\ \partial_s\left(\partial_\mu X^\nu W_\nu(X)\right)\ \underset{\rightarrow}{e}^{-ig\int_0^s W_x} + \\
&\quad + \int_0^1 ds\ \underset{\rightarrow}{e}^{-ig\int_s^1 W_x}\ \{-ig\partial_s X^\nu \partial_\mu X^\rho [W_\rho, W_\nu] + \partial_s X^\nu \partial_\mu X^\rho G_{\rho\nu}\}\ \underset{\rightarrow}{e}^{-ig\int_0^s W_x} \\
&= \int_0^1 ds \partial_s \left(\underset{\rightarrow}{e}^{-ig\int_s^1 W_x}\ \partial_\mu X^\nu W_\nu(X)\ \underset{\rightarrow}{e}^{-ig\int_0^s W_x}\right) - \\
&\quad - \int_0^1 ds\ \underset{\rightarrow}{e}^{-ig\int_s^1 W_x}\ (ig\partial_s X^\nu W_\nu(X))\partial_\mu X^\rho W_\rho(X)\ \underset{\rightarrow}{e}^{-ig\int_0^s W_x} + \qquad \text{(D.7)} \\
&\quad + \int_0^1 ds\ \underset{\rightarrow}{e}^{-ig\int_s^1 W_x}\ \partial_\mu X^\rho W_\rho(X)(ig\partial_s X^\nu W_\nu(X))\ \underset{\rightarrow}{e}^{-ig\int_0^s W_x} + \qquad \text{(D.8)} \\
&\quad + \int_0^1 ds\ \underset{\rightarrow}{e}^{-ig\int_s^1 W_x}\ \{-ig\partial_s X^\nu \partial_\mu X^\rho [W_\rho, W_\nu] + \partial_s X^\nu \partial_\mu X^\rho G_{\rho\nu}\}\ \underset{\rightarrow}{e}^{-ig\int_0^s W_x} \\
&= W_\mu(x)\ \underset{\rightarrow}{e}^{-ig\int_0^1 W_x} - 0 + \int_0^1 ds\ \underset{\rightarrow}{e}^{-ig\int_s^1 W_x}\ \partial_s X^\nu \partial_\mu X^\rho G_{\rho\nu}\ \underset{\rightarrow}{e}^{-ig\int_0^s W_x} \\
&= W_\mu(x)\ \underset{\rightarrow}{e}^{-ig\int_0^1 W_x} - \int_0^1 ds\ \underset{\rightarrow}{e}^{-ig\int_s^1 W_x}\ \partial_s X^\nu \partial_\mu X^\rho G_{\nu\rho}\ \underset{\rightarrow}{e}^{-ig\int_0^s W_x}, \qquad \text{(D.9)}
\end{aligned}$$

where eq. D.7 and eq. D.8 make use of eq. D.2. The result of eq. D.9 is nothing but the non-abelian equivalent of eq. B.4, and it can be interesting to rewrite it as follows:

$$W_\mu(x) = \int_0^1 ds\ \underset{\rightarrow}{e}^{-ig\int_s^1 W_x}\ \partial_s X^\nu \partial_\mu X^\rho G_{\nu\rho}\ \underset{\rightarrow}{e}^{-ig\int_0^s W_x} \left(\underset{\rightarrow}{e}^{-ig\int_0^1 W_x}\right)^{-1} + \frac{i}{g}\partial_\mu \Phi \Phi^{-1}. \qquad \text{(D.10)}$$

This expression gives $W_\mu(x)$ as a gauge equivalent of (see eq. C.1):

$$W'_\mu = \left(\underset{\rightarrow}{e}^{-ig\int_0^1 W_x}\right)^{-1} \int_0^1 ds\ \underset{\rightarrow}{e}^{-ig\int_s^1 W_x}\ \partial_s X^\nu \partial_\mu X^\rho G_{\nu\rho}\ \underset{\rightarrow}{e}^{-ig\int_0^s W_x}$$
$$= \int_0^1 ds\ \left(\underset{\rightarrow}{e}^{-ig\int_0^s W_x}\right)^{-1} \partial_s X^\nu \partial_\mu X^\rho G_{\nu\rho} \left(\underset{\rightarrow}{e}^{-ig\int_0^s W_x}\right). \tag{D.11}$$

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